Group Cohomology

Mica Li

Winter 2025

Introduction

We first begin by reviewing a few definitions. Throughout these notes, G will denote a group. A **(left) G-module** is an abelian group on which G acts on by additive maps on the left. Let $\mathrm{Hom}_G(A, B)$ be the set of maps from A to B. This gives us a category G-**mod** of left G-modules. [1] [2].

[1] This category can be identified with the category $\mathbb{Z}G$-**mod** over the integral group ring $\mathbb{Z}G$.

[2] It can also be identified with the functor category $\mathbf{Ab}^G$ of functors from the category G to the category **Ab** of abelian groups

Definition 1. *A **trivial G-module** is an abelian group A of G that acts trivially. This means that for all $g \in G$ and $a \in A$*

$$ga = a$$

If we consider an abelian group as a trivial $G-$module this gives us an exact functor from **Ab** to G-**mod**.

Definition 2. *Let A be a G-**mod**. Then the submodule of **fixed points** is given by*

$$A^G = \{a \in A : ga = a \text{ for all } x \in G\}$$

Then, one can see that A^G is a trivial G-module. Going further, it is the unique maximal G-trivial submodule of A.

Definition 3. *Suppose that $\varphi : A \to B$ is a G-map and let $a \in A^G$. Then, we have that since φ is a $G-$map, $\varphi(ax) = x\varphi(a) = \varphi(a)$, meaning that $\varphi(a) \in B^G$. Define $\varphi^G := \varphi|_{A^G}$. The **fixed-point functor** $Fix^G :_{\mathbb{Z}G}$ **Mod** $\to_{\mathbb{Z}G}$ **Mod** is defined by $Fix^G(A) = A^G$ and $Fix^G(\varphi) = \varphi^G$.*

Thus, one can see that Fix^G is an additive functor.

Proposition 1. *If $\mathbb{Z}$ is viewed as a $G-$trivial module, then*

$$Fix^G \cong Hom_G(\mathbb{Z}, -)$$

3

[3] In particular, this implies that Fix^G is left exact.

Proof. We define the following map

$$\tau_A : \mathrm{Hom}_A(\mathbb{Z}, A) \to A^G \qquad f \mapsto f(1)$$

Now, we check that $f(1) \in B^G$. Suppose that $x \in G$, then this means that $x(f(1)) = f(1 \cdot x) = f(1)$ since $\mathbb{Z}$ is $G-$trivial. Now it remains to show that τ_A is an isomorphism. To do this, we show that τ_A has an inverse. Let $a \in A^G$. Then, there will exist a $\mathbb{Z}-$map, f_a such that

$f_a(1) = a$. Note that we have that $xa = a$ for all $x \in G$, meaning that we have a well-defined $G-$map and f_a gives us our inverse to τ_A.

$$\begin{array}{ccc} \mathrm{Hom}_G(\mathbb{Z}, A) & \xrightarrow{\tau_A} & A^G \\ \Big\downarrow \varphi^* & & \Big\downarrow \varphi^G \\ \mathrm{Hom}_G(\mathbb{Z}, B) & \xrightarrow{\tau_B} & B^G \end{array}$$

4 □

4 Indeed, the given diagram commutes.

Definitions and Examples

Now that we have defined the fixed-point functor, we are ready to define the cohomology groups.

Definition 4. *Suppose that G is a group and A is a $G-$module. Then, the* ***cohomology groups*** *of G with coefficients in A are*

$$H^n(G, K) := Ext^n_{\mathbb{Z}G}(\mathbb{Z}, K)$$

5 *where $\mathbb{Z}$ is viewed as a trivial $G-$module.* 6

5 From this definition we have that $H^0(G, A) = A^G$

6 Note that we have defined H^n as the right derived functors of Fix^G. To review these definitions, refer to the notes in Week 7.

Given this definition we now give some examples.

Example 1. *If $G = 1$ is the trivial group, $A^G = 1$. Given that the higher derived functors of an exact functor vanishes we have that $H^n(1, A) = 0$ for $n \neq 0$.* 7

7 In some textbooks, the n is denoted by $*$

We will be going over a non-trivial example later on.

Since $\mathrm{Ext}(\mathbb{Z}, -)$ is given by the $G-$ free projective resolution of $\mathbb{Z}$ we want to start our understanding by mapping $\mathbb{Z}G \to G$.

Proposition 2. *There is a G-exact sequence*

$$0 \longrightarrow \mathcal{G} \longrightarrow \mathbb{Z}G \xrightarrow{\epsilon} \mathbb{Z} \longrightarrow 0$$

where $\epsilon : \mathbb{Z}G \to \mathbb{Z}$ is given by $\sum_{x \in G} m_x x \mapsto \sum_{x\ inG} m_x$. This is a ring map and a G-map, and $\ker \epsilon = \mathcal{G}$ is a two-sided ideal in $\mathbb{Z}G$.

Proof. Take the functor F : Groups $\to$ Rings assigning each group G an integral group ring $\mathbb{Z}G$. Then, the trivial group homomorphism $\varphi : G \to \{1\}$ induces a ring map $F\varphi : \mathbb{Z}G \to \mathbb{Z}\{1\} = $. In particular, this implies that $F\varphi = \epsilon : \sum m_x x \mapsto \sum m_x$. since ϵ is a ring homomorphism, this gives us the desired result. 8 □

8 This functor is defined by assigning to each group G its integral group ring $\mathbb{Z}G$ and to each group homomorphism $\varphi : G \to H$ the ring homomorphism $F(\varphi) : \mathbb{Z}G \to \mathbb{Z}H$, defined by $\sum m_x x \mapsto \sum m_x \varphi(x)$

Definition 5. *The map $\epsilon : \mathbb{Z}G \to \mathbb{Z}$ given by $\sum m_x x \mapsto \sum m_x$ is called the* ***augmentation*** *and $\mathcal{G} = \ker \epsilon$ is called the* ***augmentation ideal****.*

[Need to fill out this section a little more]

Cohomology Group of a Finite Cyclic Group

We are now well-equipped to calculate the cohomology groups of a finite cyclic group.

Lemma 1. *Let $G = \langle x \rangle$ be a finite cyclic group of order k and let $D = x - 1$ and $N = 1 + x + x^2 + ... + x^{k-1}$. Then*

$$\longrightarrow \mathbb{Z}G \xrightarrow{D} \mathbb{Z}G \xrightarrow{N} \mathbb{Z}G \xrightarrow{D} \mathbb{Z}G \xrightarrow{\epsilon} \mathbb{Z} \longrightarrow 0$$

is a G-free resolution of $\mathbb{Z}$ where the maps alternated between multiplication by D and multiplication by N.

Proof. Given that $\mathbb{Z}$ is commutative, we have that D and N are G-maps. To show that this gives us a G-free resolution, we want to show that it is a complex and that it is exact. We first show that it is a complex. To do so note that $DN = ND = x^k - 1 = 0$. Letting $u \in \mathbb{Z}G$ we have that

$$\epsilon(Du) = \epsilon((x-1)u) = \epsilon(x-1)\epsilon(u) = 0$$

and so we have a complex. Now all that remains is to show that this is exact. We know that ϵ is surjective. Suppose that $\ker \epsilon = \mathcal{G} = \mathrm{im} D$. So at the zeroth step, we have exactness. We first show that $\ker D \subseteq \mathrm{im} N$. If $u = \sum_{i=0}^{k-1} m_i x^i$, then

$$(x-1)u = (m_{k-1} - m_0) + (m_0 - m_1)x + ... + (m_{k-2} - m_{k-1})x^{k-1}$$

If $u \in \ker D$, then $Du = (x-1)u = 0$, and $m_{k-1} = m_0 = m_1 ... = m_{k-2}$. Therefore, $u = m_0 N \in \mathrm{im} N$. Then, we want to show that $\ker N \subseteq \mathrm{im} D$. If $u = \sum_{i=0}^{k-1} m_i x^i \in \ker N$, then $0 = \epsilon(Nu) = \epsilon(N)\epsilon(u) = k\epsilon(u)$, so $\epsilon(u) = \sum_{i=0}^{k-1} m_i = 0$ so that

$$u = -D(m_0 + (m_0 + m_1)x + ... + (m_0 + ... + m_{k-1})x^{k-1}) \in \mathrm{im} D$$

□

Theorem 1. *Let G be a finite cyclic group. If A is a G-module, define ${}_N A = \{a \in A : Na = 0\}$. Then, for all $n \geq 1$,*

$$H^0(G, A) = A^G$$
$$H^{2n-1}(G, A) =_N A/DA$$
$$H^{2n}(G, A) = A^G/NA$$

Proof. The idea of this proof lies in applying $\mathrm{Hom}_G(\mathbb{Z}, -)$ to the resolution in the previous Lemma and take homology. In more detail, this means that if $d_{2n+1} = D$ and $d_{2n} = N$ for $n \geq 0$. Then

$$\ker N^* =_N A \quad \mathrm{im} N^* = NA \quad \ker D^* = A^G \quad \mathrm{im} D^* = DA$$

where N^* and D^* are the induced maps. The formulas follow from the definition $H^m(G, A) = \ker d^*_{m+1} / \mathrm{im} d^* m$. □